Gravitation and Electrostatics Mock Exam

10 MCQ 2 FRQ - 60 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 9.8$ N/kg throughout this contest.
- Test under standard conditions, which means that you must complete the test in 60 minutes in one sitting.
- This test contains 10 Multiple Choice Questions and 2 Free Response Questions.
- Correct answers will be awarded the points shown; leaving an answer blank will be awarded zero points. The amount of partial credit is up to your discretion as the grader. Our recommendation is to be more strict than necessary.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a graphing calculator. Calculators may not be shared. Cell phones may not be used during the exam. You may not use tables, books, or formula collections.
- The number in **red** next to each question represents the amount of points the question is worth. There are a total of 40 points on this test.
- If you have any questions or clarifications, please contact us at tjhsstphysicsteam@gmail.com.

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Good luck and have fun! Turn the page to start the mock.

10 MCQs

[2] **Problem 1.** An object of mass m is placed at the center of a ring of mass M and radius R. It's then displaced a distance $d \ll R$ in the direction perpendicular to the plane of the ring. Find the frequency of oscillation ω.

a) $\omega = \sqrt{\frac{GM}{R^3}}$

b) $\omega = \sqrt{\frac{GM}{2R^3}}$

c) $\omega = \sqrt{\frac{2GM}{3R^3}}$

d) $\omega = \sqrt{\frac{3GM}{2R^3}}$

e) Oscillations do not happen

Solution. This problem can be solved with either energy or forces

1. **Forces**: Take the gravitational force of an angle $\mathrm{d}\theta$:

$$F_g = -\frac{Gm\lambda R\,\mathrm{d}\theta}{R^2 + d^2}$$

where $\lambda = M/2\pi R$. Note that the radial component of gravity will cancel out with the part of the ring 180° away, so only the vertical component $F_g \sin\theta = F_g \frac{d}{\sqrt{R^2 + d^2}}$ will contribute. Integrating this from 0 to 2π gives:

$$\begin{aligned} F_{g,\text{vertical}} &= -\frac{Gm\lambda Rd}{(R^2 + d^2)^{3/2}} \int_0^{2\pi} \mathrm{d}\theta \\ &= -\frac{Gm(\lambda\, 2\pi R)d}{(R^2 + d^2)^{3/2}} \\ &= -\frac{GmM}{(R^2 + d^2)^{3/2}} \end{aligned}$$

Since $d \ll R$, we have $(R^2 + d^2)^{3/2} \approx R^3$ and this takes the form of an oscillation with

$$\omega = \sqrt{\frac{GM}{R^3}}$$

So the answer is $\boxed{A}$

2. **Energy**: The frequency of oscillation is given by $\omega = \sqrt{\frac{U''}{m}}$. The potential energy is just

$$\begin{aligned} U &= -\frac{GmM}{\sqrt{R^2+d^2}} \\ &= -\frac{GmM}{R\sqrt{1+d^2/R^2}} \\ &\approx -\frac{GmM}{R}\left(1-\frac{d^2}{2R^2}\right) \end{aligned}$$

Where the last step comes from the binomial approximation $(1+\epsilon)^{-1/2} \approx 1 - \frac{1}{2}\epsilon$. Taking two derivatives with respect to d gives,

$$U'' = \frac{GmM}{R^3}$$

which implies that $\omega = \sqrt{GM/R^3}$

[2] **Problem 2.** Eric is in a spaceship, which orbits the sun in an elliptical orbit. His orbit has a maximum velocity of v, and a period of T. Eric then watches a NASCAR races and decides to speed up to a maximum velocity of $2v$, taking care to keep the semi major axis of his orbit the same. What is his new period T_f?

a) $T_f = T/8$

b) $T_f = T/4$

c) $T_f = T/2$

d) $T_f = T$

e) $T_f = 2T$

Solution. By Keplers 3rd law, $T^2 \propto a^3$. Since the semi major of Eric's orbit is the same, the period is the same so $T_f = T$. $\boxed{D}$

[2] **Problem 3.** A particle with mass m is initially at rest a very large (essentially infinite) distance away from another particle with mass M. What is the relative speed when they are a distance R away?

a) $M\sqrt{\frac{2G}{Rm}}$

b) $m\sqrt{\frac{2G}{RM}}$

c) $m\sqrt{\dfrac{2Gm}{RM^2}}$

d) $M\sqrt{\dfrac{2GM}{Rm^2}}$

e) $(M+m)\sqrt{\dfrac{2G}{R(m+M)}}$

Solution. Note that both energy and momentum are conserved. By momentum, we have $mv = MV \implies V = mv/M$. Energy gives us

$$0 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 - \frac{GmM}{r}$$

Substituting for V and solving the system gives us

$$v = M\sqrt{\frac{2G}{R(m+M)}}$$
$$V = m\sqrt{\frac{2G}{R(m+M)}}$$

The relative velocity is then

$$v + V = (m+M)\sqrt{\frac{2G}{R(m+M)}}$$

so our answer is $\boxed{E}$

[2] **Problem 4.** A mass is located off-center in the interior of a uniform ring of mass M. What is the direction of the gravitational force on m due to the ring?

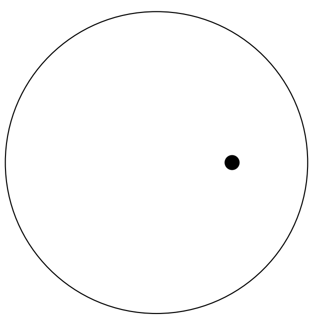


Figure 1: Question 4

a) leftward

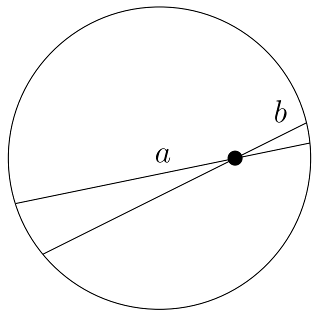
b) rightward

c) upward

d) downward

e) The force is zero inside the ring

Solution. Consider the following diagram:



If the ring has a density λ, then the gravitational attraction from the sectors is

$$\begin{aligned} F_{\text{net}} &= F_b - F_a \\ &= \frac{Gm(\lambda b\theta)}{b^2} - \frac{Gm(\lambda a\theta)}{a^2} \\ &= Gm\lambda\theta\left(\frac{1}{b} - \frac{1}{a}\right) \end{aligned}$$

where the rightward direction has been taken as positive. From the diagram, we have $b < a \implies 1/b > 1/a$, so the net force will be directed rightward. $\boxed{B}$

[2] **Problem 5.** A planet is orbiting a star in a circular orbit of radius r_0. Over a very long period of time, much greater than the period of the orbit, the star slowly and steadily loses 1% of its mass. Throughout the process, the planet's orbit remains approximately circular. The final orbit radius is closest to...?

a) $1.02r_0$

b) $1.01r_0$

c) r_0

d) $0.99r_0$

e) $0.98r_0$

Solution. Because it travels in a circle, we have

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

Note that the angular momentum $L = mvr$ is also conserved. Therefore, we have

$$M \propto v^2 r \propto \frac{L^2}{r}$$

so a 1% decrease in mass results in a 1% increase in orbit radius. $\boxed{B}$

[2] **Problem 6.** A small sphere of mass $m = 1.12\text{mg}$ and charge $q = 19.7\text{nC}$ hangs from a rope under earths gravitational field. The rope is attached to a vertical, large, uniformly charged, nonconducting sheet and makes an angle $\theta = 27.4°$ with the sheet. What is the charge density σ of the sheet?

a) $4.02 \cdot 10^{-9} \frac{\text{C}}{\text{m}^2}$

b) $5.11 \cdot 10^{-9} \frac{\text{C}}{\text{m}^2}$

c) $6.07 \cdot 10^{-9} \frac{\text{C}}{\text{m}^2}$

d) $4.72 \cdot 10^{-8} \frac{\text{C}}{\text{m}^2}$

e) $5.00 \cdot 10^{-8} \frac{\text{C}}{\text{m}^2}$

Solution. The sphere is at rest, so the net force is zero. Therefore, we have from $F = ma$:

$$T \cos\theta = mg$$
$$T \sin\theta = qE = \frac{\sigma}{2\epsilon_0} q$$

Dividing the equations and solving for σ gives:

$$\sigma = \frac{2\epsilon_0 mg \tan\theta}{q} = 5.11 \cdot 10^{-9} \frac{\text{C}}{\text{m}^2}$$

$\boxed{B}$

[2] **Problem 7.** A spherical shell is made of a thin sheet of material with charge density σ. Consider two points P_1 and P_2, which are close to each other, but just in and out of the shell, respectively. If the electric field at these points are $\mathbf{E}_1$ and $\mathbf{E}_2$ respectively, what is the value of $|\mathbf{E}_1 - \mathbf{E}_2|$?

a) $\frac{\sigma}{4\epsilon_0}$

b) $\frac{\sigma}{3\epsilon_0}$

c) $\frac{\sigma}{2\epsilon_0}$

d) $\frac{\sigma}{\epsilon_0}$

e) $\frac{2\sigma}{\epsilon_0}$

Solution. By the shell theorem, $E_1 = 0$. The remaining electric field is given by

$$E_2 = \frac{Q}{4\pi\epsilon_0 R^2}$$

By the definition of density we have $Q = 4\pi R^2\sigma$ so

$$E_2 = \frac{4\pi R^2\sigma}{4\pi R^2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

So $|E_1 - E_2| = \sigma/\epsilon_0$ or $\boxed{D}$

[2] **Problem 8.** What is the electric field on the surface of a sphere of radius R that carries a charge density proportional to the radius $\rho = \kappa r$, for a constant κ?

a) $\dfrac{\kappa R^2}{4\epsilon_0}$

b) $\dfrac{\kappa R^2}{2\epsilon_0}$

c) $\dfrac{\kappa R^2}{\epsilon_0}$

d) $\dfrac{2\kappa R^2}{\epsilon_0}$

e) $\dfrac{4\kappa R^2}{\epsilon_0}$

Solution. By Gauss's law, we have

$$E(4\pi R^2) = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Since $\mathrm{d}Q = \rho(4\pi r^2\,\mathrm{d}r) = 4\kappa\pi r^3\,\mathrm{d}r$, we have that

$$Q_{\text{enc}} = 4\kappa\pi\int_0^R r^3\,\mathrm{d}r = \kappa\pi R^4$$

Substituting back in we get

$$E(r) = \frac{\kappa R^2}{4\epsilon_0}$$

$\boxed{A}$

[2] **Problem 9.** A pair of masses with charge $-q$ and mass m are diametrically opposite a point charge Q at a radius R. The masses are then given a tangential velocity v and begin to move in a circle of radius R. What should R be such that this is possible?

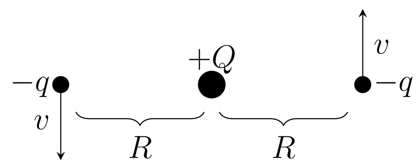


Figure 2: Problem 9

a) $R = \dfrac{kq(Q-q)}{mv^2}$

b) $R = \dfrac{kq(4Q-q)}{4mv^2}$

c) $R = \dfrac{kQ(Q-q)}{mv^2}$

d) $R = \dfrac{kQ(4Q-q)}{4mv^2}$

e) $R = \dfrac{kqQ}{mv^2}$

Solution. Writing out $F = ma$ gives:

$$k\frac{qQ}{R^2} - k\frac{q^2}{(2R)^2} = \frac{mv^2}{R}$$
$$\implies kq(4Q-q) = 4mv^2R$$

So,

$$R = \frac{kq(4Q-q)}{4mv^2}$$

$\boxed{B}$

[2] **Problem 10.** A point of charge $+q$ and mass m is stuck a distance R between two infinite lines of charge density $+\lambda$. The charge is then displaced a distance $x \ll R$ towards the left line, and begins to oscillate with a period T. Find T.

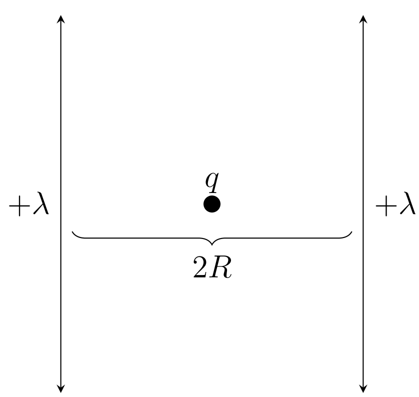


Figure 3: Question 10

a) $T = \frac{\pi R}{4}\sqrt{\frac{\pi m \epsilon_0}{q\lambda}}$

b) $T = \frac{\pi R}{2}\sqrt{\frac{\pi m \epsilon_0}{q\lambda}}$

c) $T = \pi R\sqrt{\frac{\pi m \epsilon_0}{q\lambda}}$

d) $T = 2\pi R\sqrt{\frac{\pi m \epsilon_0}{q\lambda}}$

e) $T = 4\pi R\sqrt{\frac{\pi m \epsilon_0}{q\lambda}}$

Solution. The electric field of an infinite line a distance r away is (by Gauss's law)

$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0} \implies E = \frac{\lambda}{2\pi\epsilon_0 r}$$

Now let the positive direction be the left. By the principle of superposition, we have

$$\begin{aligned} E_1 + E_2 &= \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{R+x} - \frac{1}{R-x}\right) \\ &= -\frac{\lambda}{\pi\epsilon_0}\left(\frac{x}{R^2 - x^2}\right) \\ &\approx -\frac{\lambda}{\pi\epsilon_0 R^2}x \end{aligned}$$

where the last line is from the $x \ll R$ condition. The net force is then

$$\begin{aligned} F = q(E_1 + E_2) &= -\frac{q\lambda}{\pi\epsilon_0 R^2}x = ma \\ \implies a &= -\frac{q\lambda}{\pi m\epsilon_0 R^2}x \end{aligned}$$

This is the equation of a harmonic oscillator with

$$\omega = \sqrt{\frac{q\lambda}{\pi m\epsilon_0 R^2}}$$

The period is related to ω by $T = 2\pi/\omega$, so we end up with our result of

$$T = 2\pi R\sqrt{\frac{\pi m\epsilon_0}{q\lambda}}$$

$\boxed{D}$

Free Response Questions

[10] **Problem 1.** Since gravitational and electrostatic fields both obey inverse square laws (Coulomb's law for electrostatics and Newton's Law of Universal Gravitation for gravity), there exists a Gauss's Law for Gravitation of the form

$$\oint \vec{g} \cdot d\vec{A} = \frac{m}{k}$$

where k is some constant of proportionality (similar to ϵ_0). You may use this equation without proof in this problem but you will end up needing to evaluate k for part B)

A) Suppose there is a spherical planet of radius R and mass m. Find the gravitational flux through a spherical Gaussian surface of radius r, assuming $r > R$. Express your answer in terms of m, R, r and/or any fundamental constants. Your answer should not include k.

B) Use your result from part A to find the constant of proportionality k in terms of fundamental constants.

C) Suppose the Earth is a flat disk with density ρ, radius R, and thickness t. Find the magnitude of the gravitational field on the surface of the Earth at the center of this disk. You may assume that $R \gg t$. Express your answer in terms of R, t, ρ, and/or any fundamental constants. (Hint: Use Gauss's Law and your answer for part B)

D) Using your formula from part C, evaluate numerically the magnitude of the gravitational field if $R = 6370$ km, $t = 1$ km, and $g = 9.81\ m/s^2$. (Hint: your answer should be unreasonably large for a density)

E) Now suppose that this is Minecraft and the Earth is in the shape of a cube with side length L and density ρ. Use Gauss's Law for Gravitation and a symmetry argument to find the gravitational flux through *one face of the Earth*. Express your answer in terms of L, ρ, and/or any fundamental constants.

Solution. A) By Newton's Law of Universal Gravitation, the gravitational field at a distance r from the planet is $\frac{Gm}{r^2}$. By symmetry, the gravitational field always points radially outward and is constant, so the flux is

$$\oint g dA = gA = \frac{Gm}{r^2}(4\pi r^2) = \boxed{4\pi Gm}$$

B) The gravitational flux equals $\frac{m}{k}$ and is $4\pi Gm$ as found in part A. Thus, $\frac{m}{k} = 4\pi Gm$, so $k = \boxed{\frac{1}{4\pi G}}$

C) Draw a small cylindrical Gaussian surface with caps of area A centered at the center of the disk such that the caps are on opposite sides of the Earth disk. By symmetry, and due to the fact that the Earth disk is very large compared to the thickness or any of the dimensions of our Gaussian surface, the gravitational field is constant everywhere on the cylinder and points perpendicular to the disk. The flux through the cylinder (making sure to count the flux through both caps) is therefore $\Phi = 2\oint \vec{g} \cdot d\vec{A} = 2gA$. By Gauss's Law for Gravitation, this value equals $\frac{m}{k} = 4\pi G m = 4\pi G \rho A t$. Solving for g,

$$2gA = 4\pi G \rho A t$$

$$g = \boxed{2\pi G \rho t}$$

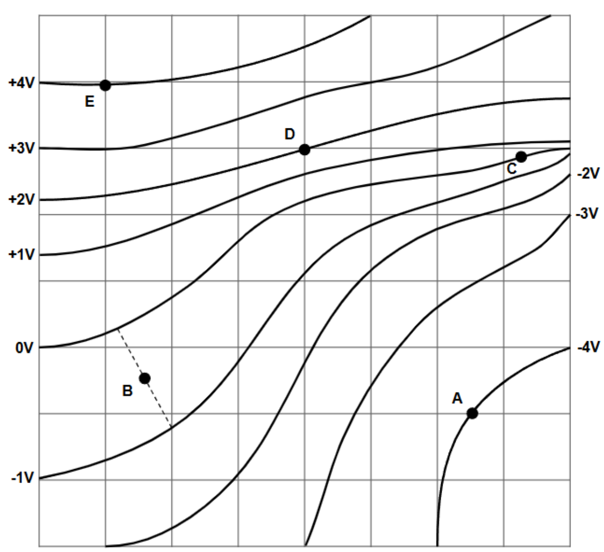
D) Using our answer from part C and solving for ρ,

$$\rho = \frac{g}{2\pi G t} = \frac{9.81 \text{ m/s}^2}{2\pi(1000 \text{ m})(6.67 * 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2})} = \boxed{2.34 * 10^7 \text{ kg/m}^3}$$

which is far denser than any known material on Earth.

E) By symmetry, the gravitational flux through every face is equal, so drawing a Gaussian surface with the same dimensions as the cube itself, the flux through one face is $\Phi = \frac{\Phi_{\text{total}}}{6}$. By Gauss's Law for Gravitation, $\Phi_{\text{total}} = \frac{m}{k} = 4\pi G \rho L^3$, so $\Phi = \boxed{\frac{2\pi G \rho L^3}{3}}$

[10] **Problem 2.** The following shows equipotential lines from an electric field with multiple charged objects not present within the diagram. Assume the scale is 1:1, with each grid cell representing a 1 cm by 1 cm square. Any points not on an equipotential line should be approximated by eyeball. Reasonable error may occur, but such will be disregarded.



A) Estimate the magnitude of the electric field at point B using the provided sketch of an electric field line.

B) At what point on the diagram is the magnitude of the electric field greatest? Justify your answer.

C) A proton is placed at point D and moves to point A. Calculate the work done by the field on the proton.

D) An electron is released from point B with an initial kinetic energy of 2 eV. Find its final kinetic energy once it reaches point E.

E) Suppose a battery consumes 5 miliwatts of power to move charge from point D to point A. Assuming that all of this power goes towards moving charge, the potentials of points A and D remain approximately constant through the entire process, and no energy is dissipated by heat, how much charge will the battery move in 1 minute?

Solution. A) We will use $|E| = |\frac{-\Delta V}{\Delta S}|$. Estimating the electric field line as the hypotenuse of a triangle with legs of 0.8 cm and 1.4 cm, we approximate $\Delta S \approx 1.612$ cm. Given $\Delta V = 1\text{V}$, we have $|E| = |\frac{-1\text{V}}{0.01612\text{m}}| = \boxed{62.0 \text{ V/m}}$

B) Once again, we use $|E| = -\frac{dV}{dS}$. We claim that the magnitude of the electric field is greatest where the equipotential lines are most densely packed - where the dS value is smallest for a given dV. Therefore, our answer is $\boxed{\text{Point C}}$.

C) We use $W = -\Delta U_E = -q\Delta V$. The charge of a proton is +e and ΔV can be quickly calculated to -6V, so we have $W = -(+\text{e})(-6\text{V}) = \boxed{6 \text{ eV}}$

Given that the proton moves with the electric field from high potential to low potential, we know that the work done should be positive, so this checks out.

D) An electron-volt (eV) is a unit of energy equal to the absolute value of the work required to move an electron over a potential difference of 1 Volt, so

$$1 \text{ eV} = \left|(-1.6 * 10^{-19}\text{C})(1 \text{ Volt})\right| = 1.6 * 10^{-19} \text{ J}$$

The electron in the problem moves through a potential difference of 4.5 Volts and its charge is -e, so the change in potential energy of the electron is -4.5 eV. Since the total energy is constant, $\Delta K = -\Delta U = 4.5$ eV. Since the intial kinetic energy is 2 eV, the final kinetic energy is $\boxed{6.5 \ \text{eV}}$

Note: The electric potential energy of a negative charge decreases when moving from low to high potential.

E) The amount of energy the battery uses to move a charge dq through the voltage difference is $dE = \Delta V dq$ so $\Delta E = Q\Delta V$. All of this energy is supplied by the battery, since the battery does work on the charges, so $\Delta E = Q\Delta V = W = Pt$, where P is power and t is the total amount of time. Therefore, $Q = \frac{Pt}{\Delta V} = \frac{(5mW)(60s)}{-6} = \boxed{-50 \text{ mC}}$
Note: this battery moves negative charges through a negative potential difference, so it increases the potential energy of the charges and therefore does positive work.