Rigid Body Motion

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1 Angular Momentum

1.1 Prerequisite Ideas

The core idea I want to impart today is the similarities of angular momentum analysis to the analysis of vectors and uniform circular motion.

It is important to remember several things throughout this analysis:

1. $\vec{\omega}$ is defined to point along the axis of rotation, as defined by the right hand rule (anticlockwise points out of the plane and is positive, and vice versa)

2. The relationship $\vec{\tau} = \mathrm{d}\vec{L}/\mathrm{d}t$ is a vector relationship, meaning that torques don't have to change the magnitude of angular momentum and can instead *change the direction of angular momentum*.

1.2 The Rotating Skew Rod

Figure 1 shows many interesting properties in regards to angular momentum and torque. Obviously one can calculate $\mathbf{L}$ and τ by the typical methods of $\sum \mathbf{r} \times \mathbf{p}$ and $\sum \mathbf{r} \times \mathbf{F}$, but in order to emphasize the vector natures of ω and τ we shall solve for them in a different manner.

In Figure 2, we see we can decompose ω into components parallel and perpendicular to the rod. The component parallel to the rod contributes no angular momentum (the $\sin\theta$ component of the cross product is 0), and as such the angu-

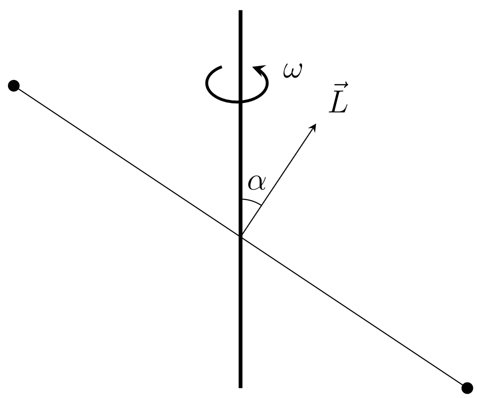

Figure 1: The rotating skew rod

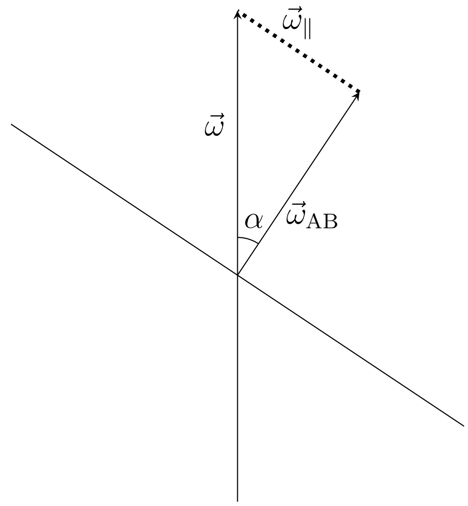

Figure 2: ω on a skew rod

lae momentum can be written as

$$\begin{aligned} |\mathbf{L}| &= I\omega_{\text{AB}} \\ &= 2m\ell^2\omega_{\text{AB}} \\ &= 2m\ell^2\omega\cos\alpha \end{aligned}$$

Now we would move on to calculating the torque, but first we would like to make some observations. The thing to keep in mind here is that, as we just saw, the *magnitude* of the angular momentum is constant of time; however, the horizontal component is executing circular motion and the vertical component is constant (see Figure 4)

> **Idea**: The vector relationship $\Delta\mathbf{L} = \tau\Delta t$ shows that external torques don't have to change the angular velocity of the object, but can instead change the direction of the angular momentum.

Now we will calculate the torque using $\tau = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$. Take a look at Figure 4. Note that we only have to consider L_h since L_z is constant with time, and will disappear in the derivative.

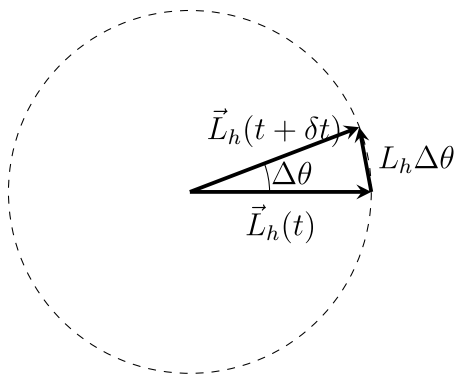

Figure 3: A vector identity

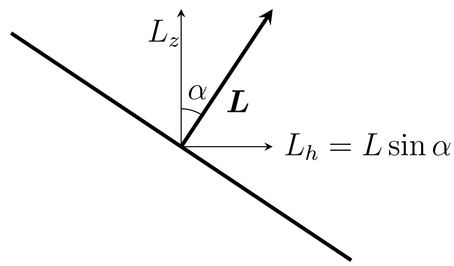

Figure 4: **L** on a skew rod

Consider a time like in figure 3. For $\Delta\theta \ll 1$, we have that $\Delta L_h \approx L_h\Delta\theta$. Therefore,

$$|\tau\Delta t| = |\Delta\mathbf{L}_h| \approx L_h\Delta\theta$$
$$|\tau| = L_h\frac{\Delta\theta}{\Delta t}$$

In the $\Delta t \to 0$ limit, we then have that:

$$\left|\frac{\mathrm{d}\mathbf{L}_h}{\mathrm{d}t}\right| = \omega L_h \tag{1}$$

In simple terms, the angular velocity ω times the component of the angular momentum *perpendicular* to ω contributes to the magnitude of the net torque.

Note that this specific case has $\ddot{\alpha} = 0$, so that we can ignore the $I\ddot{\alpha}$ term in Equation 1.

2 Analyzing Precession

Consider figure 5[1]. It might seem overwhelming to tackle these at first, but we can by just utilizing the vector nature of angular momentum.

Let's start by figuring out how it would precess, by taking a look at Figure 6.

It's natural to ask why it would precess in the z direction. This arises from the fact that there is effectively a force on the AB axis perpendicular to the $AB\Omega$

[1]Source: Kleppner and Kolenkow, 1973

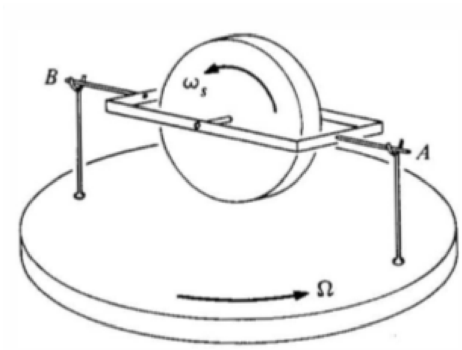

Figure 5: A Gyroscope

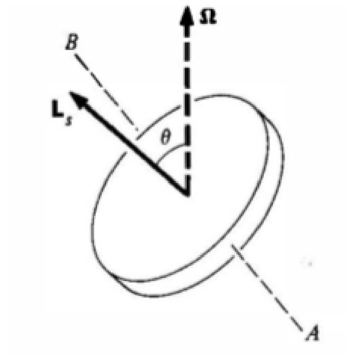

Figure 6: Precession

plane. This in turn causes a torque along the z-axis, resulting in precession like shown in Figure 6.

Now we can simply apply equation 1, taking into account the fact that θ is not constant anymore and thus must contribute to the equation. We then have:

$$\begin{aligned} \left| \frac{\mathrm{d}L_h}{\mathrm{d}t} \right| &= I_{\mathrm{AB}}\ddot{\theta} + \Omega L_s \sin\theta \\ &\approx I_{\mathrm{AB}}\ddot{\theta} + \Omega L_s \theta \end{aligned}$$

assuming $\theta \ll 1$. Note that I_{AB} is the moment of inertia of the disk about the AB axis.

Now due to the pivot at the center of the disk, there cannot be any torques along the AB axis, therefore $|\frac{\mathrm{d}L_h}{\mathrm{d}t}| = 0$. We then have:

$$I_{\mathrm{AB}}\ddot{\theta} + (\Omega L_s)\theta = 0$$

Which takes the form of simple harmonic motion. We then have our solution as:

$$\theta(t) = \theta_0 \sin(\beta t)$$

Where

$$\beta \equiv \sqrt{\frac{L_s \Omega}{I_{\mathrm{AB}}}} = \sqrt{\frac{\omega_s \Omega I_s}{I_{\mathrm{AB}}}}$$

Wow!

3 Moment of Inertia Tensor

We start with the task of trying to find the angular momentum of an object with respect to the center of mass of the object. From the definition of angular momentum we have

$$\mathbf{L} = \sum \mathbf{r} \times m\dot{\mathbf{r}}$$

Since $\mathbf{r}$ is a rotating vector we have $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$ or

$$\mathbf{L} = \sum \mathbf{r} \times m(\boldsymbol{\omega} \times \mathbf{r})$$

If we evalulate the $\boldsymbol{\omega} \times \mathbf{r}$ using $\boldsymbol{\omega} \equiv \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$ we end up with

$$\boldsymbol{\omega} \times \boldsymbol{r} = (z\omega_y - y\omega_z)\hat{\boldsymbol{\imath}} - (z\omega_x - x\omega_z)\hat{\boldsymbol{\jmath}} + (y\omega_x - x\omega_y)\hat{\boldsymbol{k}}$$

If we work with a specific component of the final angular momentum, like for example the x component, we have:

$$[\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})]_x = y(\boldsymbol{\omega} \times \boldsymbol{r})_z - z(\boldsymbol{\omega} \times \boldsymbol{r})_y$$

Substituting the previous equation into the current one, we find the angular momentum is

$$\begin{aligned} L_x &= \omega_x \sum m_j(y_j^2 + z_j^2) - \omega_y \sum m_j x_j y_j - \omega_z \sum m_j x_j y_j \\ &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \end{aligned}$$

I_{xx} is what we used to use as the definition of the moment of inertia, and what we are familiar with. I_{xy} and I_{xz} are called the products of inertia.

We can get the other components of $\mathbf{L}$ by cycling coordinates: $x \to y, y \to z, z \to x$.

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

We can write this in the more compact form

$$\mathbf{L} = I_{ij}\boldsymbol{\omega}$$

Where

$$I_{ij} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

This is called the *Moment of Inertia Tensor*.

Interestingly enough, it is always possible to choose a set of axis such that the products of inertia are zero. These axis are called the *primordial axis*, and coincide with the symmetry axes of the object. In such cases the moment of inertia tensor takes the simple(r) form

$$I_{ij} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

Interestingly, the primordial axes are the *eigenvectors* of the moment of inertia tensor.

4 Eulers Rigid Body Equations

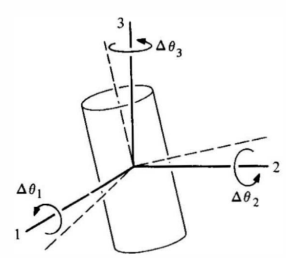

Figure 7: The Primordial Axis Rotating

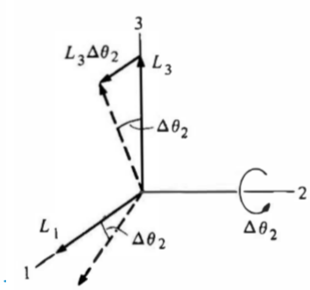

Figure 8: Rotation about 2

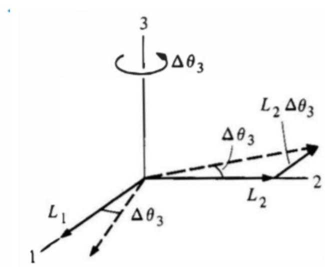

Figure 9: Rotation about 3

We now turn to finding the full form of $\mathrm{d}\boldsymbol{L}/\mathrm{dt}$. We start with one component, say $(\mathrm{d}\boldsymbol{L}/\mathrm{dt})_1$. Consider a rotation rotating about it's primordial axis at a time t. The change in L_1, $\Delta L_1 = L_1(t+\delta t) - L_1(t)$, then has several contributions.

The first contribution is obviously $I_1\omega_1$. We can get the contributions from $\Delta\theta_2$ and $\Delta\theta_3$ by looking at the components of L_2 and L_3 along axis 1. This gives contributions of $\Delta(L_1 \cos\theta_2) + \Delta(L_3 \sin\theta_2)$ from rotating about θ_2, and $\Delta(L_1 \cos\theta_3) - \Delta(L_2 \sin\theta_3)$. Making the small angle approximations for $\Delta\theta <<$ 1, we have $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. As such, the contributions are

$$L_3\Delta\theta_2 \qquad\qquad \text{and} \qquad\qquad -L_2\Delta\theta_3$$

I won't prove it rigorously, but it's possible that the contributions add algebraically. Using $L_2 = I_2\omega_2$ and $L_3 = I_3\omega_3$, we have that

$$\Delta L = I\omega_1 + I_3\omega_3\Delta\theta_2 - I_2\omega_2\Delta\theta_3$$

Dividing by Δt and taking the limit $t \to 0$ gives:

$$\tau_1 = I_1\ddot{\theta}_1 + (I_3 - I_2)\omega_2\omega_3$$

We can get the other dimensions by cycling the coordinates $1 \to 2, 2 \to 3, 3 \to 1$. As such we get our final result:

$$\begin{aligned}
\tau_1 &= I_1\ddot{\theta}_1 + (I_3 - I_2)\omega_2\omega_3 \\
\tau_2 &= I_1\ddot{\theta}_2 + (I_1 - I_3)\omega_1\omega_3 \\
\tau_3 &= I_3\ddot{\theta}_3 + (I_2 - I_1)\omega_1\omega_2
\end{aligned}$$