# **David Tong: Vector Calculus Example Sheet 3**

By Resident Physics Noob

A note on notation: I try to stick to the notation used in the lecture notes. However, based on space (and how lazy I'm feeling), I may use some shortcuts (i.e.  $\partial_x$  instead of  $\frac{\partial}{\partial x}$ , omit bounds like S or  $\partial S$ , etc.)

## 1)

#### **1.i**)

Switch to polar coordinates:

$$\begin{aligned} x &= R\cos(\phi) \\ y &= R\sin(\phi) \\ \mathrm{d}x &= -R\sin(\phi)\,\mathrm{d}\phi \\ \mathrm{d}y &= R\cos(\phi)\,\mathrm{d}\phi \end{aligned}$$

Then we have

$$\begin{split} I &= \int_0^{2\pi} \mathrm{d}\phi \big[ - \big(R^2 \cos^2 \phi\big) (R \sin \phi) (-R \sin \phi) + (R \cos \phi) \big(R^2 \sin^2 \phi\big) (R \cos \phi) \big] \\ &= \frac{\pi R^4}{2} \end{split}$$

For the area integral, we know that  $P = -x^2y$  and  $Q = xy^2$  - therefore we also have

$$\frac{\partial P}{\partial y} = -x^2$$
$$\frac{\partial Q}{\partial x} = y^2$$

Substituting into Greens theorem (and using  $dA = \rho d\rho d\phi$ ) gives

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^R \mathrm{d}\rho \,\rho(x^2 + y^2) = \frac{\pi R^4}{2}$$

in agreement with our previous result.

#### 1.ii)

By Greens theorem, we have

$$I = \int_A (x^2 + y^2) \,\mathrm{d}A$$

This integral doesn't change if the space is rotated, because  $x^2 + y^2$  is simply the distance from the origin, and dA is independent of the coordinate system.

## 2)

#### 2.i)

We have that  $\nabla \times F = -2z\hat{k}$ . Our surface d**S** is, in spherical coordinates

$$R^2 \sin(\theta) \boldsymbol{e_r} \, \mathrm{d}\theta \, \mathrm{d}\phi = \sin(\theta) \boldsymbol{e_r} \, \mathrm{d}\theta \, \mathrm{d}\phi$$

since R = 1. Using  $\hat{k} = \cos \theta$  we have

$$(\nabla \times F) \cdot \mathrm{d}\boldsymbol{S} = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi(-2z\sin\theta\cos\theta)$$
$$= -2\pi$$

The line integral is the rim of the shell, and is parameterized by  $\rho=1, z=0$  or

$$\boldsymbol{x} = (\cos \phi, \sin \phi, 0) \Longrightarrow \mathrm{d}\boldsymbol{x} = (-\sin \phi, \cos \phi, 0) \,\mathrm{d}\phi$$

Thus

$$F \cdot d\mathbf{x} = -y \sin \phi - x \cos \phi = -1$$
$$\int_{\partial S} F \cdot d\mathbf{x} = \int_{0}^{2\pi} - d\phi = -2\pi$$

3)

### 3.i)

Stokes theorem says

$$\oint_{\partial S} (\boldsymbol{a} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{x} = \int_{S} \nabla \times (\boldsymbol{a} \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

Using

$$A \cdot (B \times C) = B \cdot (C \times A)$$
$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$
$$A \times B = -B \times A$$

we have

$$\begin{split} - \boldsymbol{a} \cdot \oint_{\partial S} \mathrm{d}\boldsymbol{x} \times \boldsymbol{F} &= \int_{S} [\boldsymbol{a} (\nabla \cdot \boldsymbol{F}) - (\boldsymbol{a} \cdot \nabla) \boldsymbol{F}] \, \mathrm{d}\boldsymbol{S} \\ &= \boldsymbol{a} \int_{S} [(\nabla \cdot \boldsymbol{F}) \, \mathrm{d}\boldsymbol{S} - (\mathrm{d}\boldsymbol{S} \cdot \nabla) \boldsymbol{F}] \end{split}$$

We can rewrite

$$(\nabla \cdot \boldsymbol{F}) \, \mathrm{d} \boldsymbol{S} - (\mathrm{d} \boldsymbol{S} \cdot \nabla) \boldsymbol{F} = -((\mathrm{d} \boldsymbol{S} \times \nabla) \times \boldsymbol{F})$$

Substituting back in, and doing some algebra gives

$$\oint_{\partial S} \mathrm{d}\boldsymbol{x} \times \boldsymbol{F} = \int_{S} (\mathrm{d}\boldsymbol{S} \times \nabla) \times \boldsymbol{F}$$

as we intended to prove.

#### **3.ii**)

Let's do the line integral first.

1. From origin to (1,0,0) we have  $d\mathbf{x} = (1,0,0) dx$ , so

$$\int_0^1 \mathrm{d}\boldsymbol{x} \times \boldsymbol{F} = \int_0^1 (1,0,0) \times (x,0,0) \,\mathrm{d}x = 0$$

2. From (1,0,0) to (1,1,0) we have dx = (0,1,0) dy and F = (1, y, 0), so

$$\int_0^1 (0,1,0) \times (1,y,0) \, \mathrm{d} y = (0,0,-1)$$

3. From (1,1,0) to (0,1,0), we have  $d\mathbf{x} = (-1, 0, 0) dx$  and  $\mathbf{F} = (x, 1, 0)$ , so

$$\int_0^1 (-1,0,0) \times (x,1,0) \, \mathrm{d}x = (0,0,-1)$$

4. From (0,1,0) to origin, we have  $d\boldsymbol{x} = (0, -1, 0) dy$  and  $\boldsymbol{F} = (0, x, 0)$  so

$$\int_0^1 (0, -1, 0) \times (0, x, 0) \, \mathrm{d}y = 0$$

so the total is (0, 0, -2).

For the surface integral we have dS = (0, 0, 1) dx dy so

$$\begin{split} \mathrm{d}\boldsymbol{S} \times \nabla &= \hat{k} \times \nabla = \left( -\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0 \right) \\ \int_{S} (\mathrm{d}\boldsymbol{S} \times \nabla) \times \boldsymbol{F} &= \int_{0}^{1} \int_{0}^{1} \left( -\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, 0 \right) \times (x, y, z) \, \mathrm{d}x \, \mathrm{d}y \\ &= -2\hat{k} \end{split}$$

in agreement with our previous result.

### 4)

Because r = 1 we have that F(x) = x. We also have the normal being in the same direction as x, and because r = 1 we know that x is already unit sized. Therefore we have that

$$\int_{S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_{S} \boldsymbol{x} \cdot \boldsymbol{x} \, \mathrm{d}S = \text{Surface area of sphere} = 4\pi$$

By Stokes theorem we expect that if  $F = \nabla \times A$  we should be able to write

$$\int_{S} (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \int_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = 0$$

where the last line comes from the fact that the boundary of a closed surface (like our sphere) is the empty set. However, as calculated above, we clearly got a nonzero value.

We can calculate  $\nabla \cdot F$  with the divergence theorem. Because the boundary of a sphere is the empty set, we have that

$$\int_{V} (\nabla \cdot \boldsymbol{F}) \, \mathrm{d}V = \int_{\partial V} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0$$
$$\implies \nabla \cdot \boldsymbol{F} = 0$$

However, note that we cannot use this to conclude that  $F = \nabla \times A$  globally, because the given proof had the requirement that the domain of F(x) is  $\mathbb{R}^3$ , but in our case the domain is  $\mathbb{R}^3 \setminus 0$ .

## **5**)

I got lazy and put the curl into Wolfram Alpha, which spit out the F(x) given in the previous problem. However, note that this isn't a contradiction because the discontinuity at  $x^2 + y^2 = 0$  still exists.

TODO: I also got too lazy to do the surface integral

## 6)

**6.i) E(r)** We have

$$\int_{S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_{S} \boldsymbol{E} \cdot \mathrm{d}S = 4\pi r^{2} \boldsymbol{E}(r)$$

By Gauss' law, we know that this equals  $\frac{Q}{\varepsilon_0}$  so we have

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\int_V \rho \, \mathrm{d}V}{4\pi\varepsilon_0 r^2}$$

- From  $r \leq a$ , the charge integral is 0 so E = 0.
- For a < r < b, the charge integral is

$$\begin{split} \int_a^r \frac{\rho_0}{a} r(4\pi r^2) \, \mathrm{d}r &= \frac{\pi \rho_0}{a} (r^4 - a^4) \\ \Longrightarrow E &= \frac{\rho_0}{4a\varepsilon_0 r^2} (r^4 - a^4) \end{split}$$

• For  $r \ge b$ , the charge is simply (from the previous section)

$$\begin{split} Q &= \frac{\pi \rho_0}{a} (b^4 - a^4) \\ \Longrightarrow E &= \frac{\rho_0}{4a\varepsilon_0 r^2} (b^4 - a^4) \end{split}$$

#### **6.ii)** $\phi(r)$

It's spherically symmetric, so we test a solution of the form  $\nabla^2 r^p = p(p+1)r^{p-2}$ .

- For  $r \leq a$ , this should equal 0 so we have  $\phi = \frac{A}{r} + B$ ,  $r \leq a$ .
- For a < r < b, this should be proportional to r so we try a solution of the form

$$\kappa r^3 \Longrightarrow \nabla^2(\kappa r^3) = 12\kappa r = \rho_0 \frac{r}{a}$$
$$\Longrightarrow \kappa = \frac{\rho_0}{12a}$$

so we have

$$\phi = \frac{\rho_0}{12a}r^3 + \frac{C}{r} + D$$

• For  $r \ge b$  it should take the form  $\frac{E}{r} + F$ 

Now lets impose some conditions. As  $r \to \infty$ ,  $\phi \to 0$  so F = 0. We expect r = 0 to be nonsingular so A = 0. We also expect it to be continuous:

$$B = \frac{\rho_0 a^2}{12} + \frac{C}{a} + D$$
$$\frac{E}{b} = \frac{\rho_0 b^3}{12a} + \frac{C}{b} + D$$

Finally, we expect  $\phi'$  to be continuous at r = a and r = b:

$$\begin{split} 0 &= \frac{\rho_0 a}{4} - \frac{C}{a^2} \\ - \frac{E}{b^2} &= \frac{\rho_0 b^2}{4a} - \frac{C}{b^2} \end{split}$$

Solving gives

$$\begin{cases} B\\ C\\ D\\ E \end{cases} = \frac{\rho_0}{12a} \begin{pmatrix} 4(a^3 - b^3)\\ 3a^4\\ -4b^3\\ 3(a^4 - b^4) \end{pmatrix}$$
  
$$\Longrightarrow \phi = \begin{cases} \rho_0 \frac{a^3 - b^3}{3a} & \text{if } r < a \\ \frac{\rho_0}{12a}r^3 + \frac{\rho_0 a^3}{4r} - \frac{\rho_0 b^3}{3a} & \text{if } a < r < b \\ \frac{\rho_0(a^4 - b^4)}{4ar} & \text{if } r \ge b \end{cases}$$

You can check yourself that computing  $\nabla \phi$  gives E.

# 7)

## 7.i) Identities

We have

$$\nabla \psi = \frac{\partial \psi}{\partial x^i} e_i = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x^i} e_i = \psi'(r) \frac{x_i}{r} e_i = \psi'(r) \frac{x}{r}$$

For the Laplacian, we have

$$\begin{split} \nabla \cdot (\nabla \psi) &= \frac{\partial}{\partial x^i} \Big[ \psi'(r) \frac{x_i}{r} \Big] \\ &= \frac{\partial \psi'}{\partial x^i} \frac{x_i}{r} + \psi'(r) \frac{\partial}{\partial x^i} \Big( \frac{x_i}{r} \Big) \\ &= \psi''(r) \frac{x_i^2}{r^2} + \frac{\psi'(r)}{r} - \frac{1}{r} \psi'(r) \frac{x^{i^2}}{r^2} \\ &= \psi''(r) + \frac{3}{r} \psi'(r) - \frac{1}{r} \psi'(r) \\ &= \psi''(r) + \frac{2}{r} \psi'(r) \end{split}$$

assuming we're in  $\mathbb{R}^3$ . The Laplacian in spherical coordinates (ignoring all  $\theta$  and  $\phi$  terms) is

$$\begin{split} \nabla^2 \psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \psi'(r)) \\ &= \psi''(r) + \frac{2}{r} \psi'(r) \end{split}$$

in agreement with the result using suffix notation.

## **7.ii**) $\nabla^2 \psi = 1$

We try a solution of the form  $r^p$ , and note that the result is independent of r so we'll guess  $\kappa r^2$ .

$$\nabla^2 \bigl(\kappa r^2\bigr) = 6\kappa = 1 \Longrightarrow \kappa = \frac{1}{6}$$

so we have the general form as

$$\psi(r) = \frac{1}{6}r^2 + \frac{A}{r} + B$$

Alternatively, you could solve this to get the same solution:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = 1$$

This must be non-singular so A = 0, and we must have  $\psi(R) = 1 \Longrightarrow B = \frac{5}{6}R^2$ . Therefore our full solution is

$$\psi(r)=\frac{1}{6}r^2+\frac{5}{6}R^2$$

### 8)

8.i)  $\nabla^2 \phi = \nabla^2 \psi = 0$ We have that

$$\begin{split} \frac{1}{2} \big(\partial_x + i\partial_y\big)(\phi + i\psi) &= \frac{1}{2} \big(\partial_x \phi - \partial_y \psi + i\big(\partial_x \psi + \partial_y \phi\big)\big) \\ &= 0 \end{split}$$

For this to equal to zero, the real and imaginary parts must be zero, or

$$\begin{array}{l} \partial_x \phi = \partial_y \psi \quad \Rightarrow \partial_x^2 \phi = \partial_x \partial_y \psi \\ \\ \partial_y \phi = -\partial_x \psi \Rightarrow \partial_y^2 \phi = -\partial_y \partial_x \psi \end{array}$$

The Laplacian of  $\phi$  is simply  $\partial_x^2 \phi + \partial_y^2 \phi = \partial_x \partial_y \psi - \partial_y \partial_x \psi$ . For sufficiently smooth functions, the order of partial derivatives doesn't matter and  $\nabla^2 \phi = 0$ . A similar process holds for  $\psi$ .

#### 8.ii) $\phi = c$ perpendicular to $\psi = k$

To show the condition on the curves, we must show that  $\nabla \phi \cdot \nabla \psi = 0$ , since the gradient gives a normal vector to the curve. We have that

$$\partial_x \phi \cdot \partial_x \psi + \partial_y \phi \cdot \partial_y \psi = (\partial_y \psi) (-\partial_y \phi) + \partial_y \phi \cdot \partial_y \psi = 0$$

thus proving the claim.

#### 8.iii) imaginary shenanigans

We have  $f = (x + iy)e^{x+iy} = e^x(x + iy)e^{iy}$ . By Euler's formula, we then have that

$$f = e^x (x + iy)(\cos y + i\sin y)$$
  
=  $e^x (x\cos y - y\sin y) + ie^x (y\cos y + x\sin y)$ 

from where it is trivial to pick out  $\phi = e^x(x\cos y - y\sin y)$  and  $\psi = e^x(y\cos y + x\sin y)$ .

9)

#### 9a)

We have  $\nabla^2(f(x)e^{\alpha y}) = [f''(x) + \alpha^2 f(x)]e^{\alpha y}$ . Setting this equal to zero gives us the equation of harmonic oscillations:  $f'' + \alpha^2 f = 0$ . This means  $f(x) = A\sin(\alpha x + \phi)$ , where A and  $\phi$  are constants.

- $\psi(0, y) = 0$  means that we must have  $\sin \phi = 0 \Rightarrow \phi = 0$
- $\psi(a, y) = 0$  means that the  $\sin(\alpha a)$  term must be zero again, so  $\alpha a = \pm \pi \Rightarrow \alpha = \pm \frac{\pi}{a}$ . In our case, let's arbitrarily choose the negative solution. Then, our equation is

$$\psi(x,y) = -A\sin\left(\frac{\pi}{a}x\right)e^{-\pi y/a}$$

• Due to our choice of a negative value for  $\alpha$ , the condition

$$\lim_{y\to\infty}\psi(x,y)=0$$

is satisfied.

• Since  $\psi(x,0) = \lambda \sin(\frac{\pi}{a}x)$  we have that  $A = -\lambda$ 

Therefore, the solution to Laplace's equation for the given boundary conditions is

$$\psi(x,y) = \lambda \sin\left(\frac{\pi}{a}x\right) e^{-\pi y/a}$$

#### **9b**)

The Laplacian of  $\psi(r, \theta)$  in cylindrical coordinates is

$$abla^2\psi(r, heta)=ig(lpha^2-eta^2ig)
ho^{lpha-2}\coseta heta$$

This is a solution to the Laplacian equation iff  $\alpha = \pm \beta$ .

#### 9b.i)

Let's assume this function must be non-singular at r = 0, so we have  $\alpha = \beta$ . We have

$$\psi(R,\theta) = AR^{\beta}\cos\beta\theta = \lambda\cos\theta$$

Therefore  $\beta = 1$  and  $A = \lambda R^{-1}$  and we have

$$\psi(r,\theta) = \lambda \frac{r}{R} \cos \theta$$

9b.ii)

In order for

$$\lim_{r\to\infty}\psi(r,\theta)=0$$

we must have  $\alpha < 0$ , so we'll use  $\alpha = -\beta$ . Then the condition

 $AR^{-\beta}\cos\beta\theta=\lambda\cos\theta$ 

tells us that  $\beta = 1$  and  $A = \lambda R$  so we have

$$\psi(r,\theta) = \lambda \frac{R}{r} \cos \theta$$

#### 9b.iii)

Recall that the Laplacian solutions are linear, so we can write

$$\psi(x,y) = \left(Ar^{\beta} + Br^{-\beta}\right)\cos\beta\theta$$

We have from the Dirichlet condition that

$$\psi(b,\theta) = \left(Ab^{\beta} + Bb^{-\beta}\right)\cos\beta\theta = \lambda\cos2\theta$$

Therefore,  $\beta = 2$  and  $Ab^2 + Bb^{-2} = \lambda$ .

To apply the Neumann condition, note that the normal in polar coordinates is  $\hat{r}$ . Then we have

$$\hat{r} \cdot \nabla \psi(a, \theta) = (2Aa - 2Ba^{-3})\cos 2\theta = 0$$
$$\implies Aa^4 = B$$

Combining with  $Ab^2 + Bb^{-2} = \lambda$  we have

$$A = \frac{\lambda b^2}{b^4 + a^4}$$
$$B = \frac{\lambda b^2 a^4}{b^4 + a^4}$$

Substituting back in gives us our equation

$$\psi(r,\theta) = \left(\frac{\lambda b^2}{b^4 + a^4}r^2 + \frac{\lambda b^2 a^4}{b^4 + a^4}r^{-2}\right)\cos 2\theta$$

### 10)

We have

$$\int_{V} (\phi \nabla^{2} \psi - \psi \nabla^{2} \phi) \, \mathrm{d}V = \int_{V} \phi \nabla^{2} \psi \, \mathrm{d}V - \int_{V} \psi \nabla^{2} \phi \, \mathrm{d}V$$

By Greens first identity,

$$\begin{split} \left( \int_{\partial V} \phi \nabla \psi \cdot \mathrm{d}\boldsymbol{S} - \int_{V} (\nabla \psi) \cdot (\nabla \phi) \, \mathrm{d}V \right) - \left( \int_{\partial V} \psi \nabla \phi \cdot \mathrm{d}\boldsymbol{S} - \int_{V} (\nabla \psi) \cdot (\nabla \phi) \, \mathrm{d}V \right) \\ = \int_{\partial V} (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathrm{d}\boldsymbol{S} \end{split}$$

## 11)

Let  $\phi := \psi_1 - \psi_2$ , where  $\psi_1$  and  $\psi_2$  are two different solutions to the problem. We'll try to prove this theorem by contradiction. Note that

$$\begin{aligned} -\nabla^2 \phi + \phi &= -\nabla^2 \psi_1 + \psi_1 - (\nabla^2 \psi_2 + \psi_2) \\ &= \rho(x) - \rho(x) \\ &= 0 \end{aligned}$$
 (1)

and

$$\begin{split} \hat{n} \cdot \nabla \phi &= \hat{n} \cdot \nabla \psi_1 - \hat{n} \cdot \nabla \psi_2 \\ &= f(\boldsymbol{x}) - f(\boldsymbol{x}) \\ &= 0 \end{split}$$

Now, let's multiply Equation 1 by  $\phi$ , and integrate

$$\int_{V} (-\nabla^{2}\phi + \phi)\phi \,\mathrm{d}V = 0$$
$$\Longrightarrow - \int_{V} \phi \nabla^{2}\phi \,\mathrm{d}V + \int_{V} \phi^{2} \,\mathrm{d}V = 0$$

By Greens first identity, this equals

$$\int_{V} (\nabla \phi)^{2} \,\mathrm{d}V - \int_{\partial V} \phi \nabla \phi \cdot \mathrm{d}S + \int_{V} \phi^{2} \,\mathrm{d}V = 0$$

But since  $d\mathbf{S} = \hat{n} dS$  and  $\hat{n} \cdot \nabla \phi = 0$ , the middle term vanishes and we have

$$\int_{V} (\nabla \phi)^2 \,\mathrm{d}V + \int_{V} \phi^2 \,\mathrm{d}V = 0$$

Neither of the integrals can be < 0, so both must be zero everywhere, implying  $\phi = 0 \Rightarrow \psi_1 = \psi_2$  in direct contradiction of our initial assumption. Hence any solution to this boundary value problem must be unique.

### 12)

#### 12.i) Uniqueness

Once again, we go looking for a proof by contradiction. Let  $\phi = \psi_1 - \psi_2$ , where  $\psi_1$  and  $\psi_2$  are two different solutions to the boundary conditions. It follows that  $\nabla^2 \phi = \nabla^2 \psi_1 - \nabla^2 \psi_2 = 0$ . Then, simply multiply both sides by  $\phi$  and apply Green's first identity:

$$\begin{split} &\int_{V} \phi \nabla^{2} \phi \, \mathrm{d} V = 0 \\ \Longrightarrow &\int_{V} (\nabla \phi)^{2} \, \mathrm{d} V - \int_{\partial V} (\phi \nabla \phi) \cdot \hat{n} \, \mathrm{d} S = 0 \end{split}$$

From the boundary condition we have that

$$\begin{split} (\hat{n} \cdot \nabla \phi)g(\boldsymbol{x}) + \phi &= (\hat{n} \cdot \nabla \psi_1 - \hat{n} \cdot \nabla \psi_2)g(\boldsymbol{x}) + \psi_1 - \psi_2 \\ &= (\hat{n} \cdot \nabla \psi_1)g(\boldsymbol{x}) + \psi_1 - ((\hat{n} \cdot \nabla \psi_2)g(\boldsymbol{x}) + \psi_2) \\ &= f(\boldsymbol{x}) - f(\boldsymbol{x}) \\ &= 0 \\ \implies \hat{n} \cdot \nabla \phi = -\frac{\phi}{g(\boldsymbol{x})} \end{split}$$

Substituting this back into our integral, we get

$$\int_{V} (\nabla \phi)^2 \,\mathrm{d}V + \int_{\partial V} \frac{\phi^2}{g(\boldsymbol{x})} \,\mathrm{d}S = 0$$

We were given the condition  $g(x) \ge 0$  on  $\partial V$ , so no quantities in this expression can be negative, so they must be zero everywhere. In other words,  $\phi = 0 \Rightarrow \psi_1 = \psi_2$ , so if a solution exists, then it is unique.

#### 12.ii)

Let's take a spherical solution of the form  $\psi = \frac{A}{r} + B$ . At r = 1, the boundary condition is

$$\begin{split} (\hat{n}\cdot\nabla\psi)(-1)+\psi&=0\\ \Longrightarrow \frac{\partial\psi}{\partial r}=\psi\\ -\frac{A}{r^2}&=\frac{A}{r}+B\\ B&=-2A \end{split}$$

where in the last line we used r = 1. Therefore, we have

$$\psi(\boldsymbol{x}) = A \bigg( \frac{1}{|\boldsymbol{x}|} - 2 \bigg) \,\, \forall A \neq 0$$

TODO: this solution is singular at r = 0

## 13)

#### 13.i)

By a rearranging of Greens first identity, we have that

$$\int_V (\nabla v) \cdot (\nabla u) \, \mathrm{d} V = \int_{\partial V} v (\nabla u) \cdot \mathrm{d} \boldsymbol{S} - \int_V v \nabla^2 u \, \mathrm{d} V$$

We have that u is harmonic so  $\nabla^2 u = 0$  and the second term on the right is 0. Additionally, we have that v = 0 on  $\partial V$  so the left term on the right is also 0. Therefore, we have that

$$\int_V (\nabla v) \cdot (\nabla u) \, \mathrm{d} V = 0$$

as we set out to show.

#### 13.ii)

We have that w = v + u, so

$$|\nabla w|^2 = |\nabla v + \nabla u|^2 = |\nabla v|^2 + 2(\nabla v) \cdot (\nabla u) + |\nabla u|^2$$

Integrate both sides over a volume V to get

$$\begin{split} \int_{V} |\nabla w|^2 \, \mathrm{d}V &= \int_{V} |\nabla v|^2 \, \mathrm{d}V + 2 \int_{V} (\nabla v) \cdot (\nabla u) \, \mathrm{d}V + \int_{V} |\nabla u|^2 \, \mathrm{d}V \\ &= \int_{V} |\nabla v|^2 \, \mathrm{d}V + \int_{V} |\nabla u|^2 \, \mathrm{d}V \end{split}$$

where the last step was due to the expression we proved in part 1. Note that  $|\nabla v|^2 dV \ge 0$ always, so we can rewrite this as

$$\int_V |\nabla w|^2 \,\mathrm{d} V \geq \int_V |\nabla u|^2 \,\mathrm{d} V$$

## 14)

#### 14.i)

We have from the mean value property that  $\psi(a) = \frac{1}{4\pi r^2} \int_{\partial V} \psi(x) \, dS$ , where V is the volume of the ball. Our goal is to express  $\psi(a)$  in terms of the volume V instead of the surface  $\partial V$ .

From the definition of a volume integral, we have

$$\int_{V} \psi(\boldsymbol{x}) \, \mathrm{d}V = \int_{0}^{r} \mathrm{d}r' \int_{\partial V} \psi(\boldsymbol{x}) \, \mathrm{d}S$$

From here, notice that  $\int_{\partial V} \psi(\boldsymbol{x}) \, \mathrm{d}S = 4\pi r^2 \psi(\boldsymbol{a})$ . Plugging it in gives us

$$\begin{split} \int_{V} \psi(\boldsymbol{x}) \, \mathrm{d}V &= \int_{0}^{r} 4\pi r'^{2} \psi(\boldsymbol{a}) \, \mathrm{d}r' \\ &= \frac{4}{3}\pi r^{3} \psi(\boldsymbol{a}) \\ &\Longrightarrow \psi(\boldsymbol{a}) = \frac{1}{4\pi r^{3}/3} \int_{V} \psi(\boldsymbol{x}) \, \mathrm{d}V \end{split}$$

as we set out to show.

#### 14.ii)

We have

$$\psi(\boldsymbol{a}) = \frac{1}{V} \int_{V} \psi(\boldsymbol{x}) \, \mathrm{d} V$$

Next, take the gradient and consider the limit as  $r \to \infty$ :

$$\lim_{r \to \infty} \nabla \psi(\boldsymbol{a}) = \lim_{r \to \infty} \nabla \left( \frac{1}{V} \int_{V} \psi(\boldsymbol{x}) \, \mathrm{d}V \right)$$

Because  $\psi(\boldsymbol{x}) \leq \mathcal{M}$  for some  $\mathcal{M}$ , the average value approaches a constant as  $r \to \infty$  and thus the gradient goes to 0. This holds for arbitrary points  $\boldsymbol{a} \in \mathbb{R}^3$ .

### 15)

Consider a small amount of water with mass  $\rho \, dV$ . A time dt later, it will still have a mass  $\rho \, dV = \rho \boldsymbol{x} \cdot d\boldsymbol{S}$ , assuming water is incompressible and the density stayed approximately constant. Integrating, we then have

$$\int_{V} \rho \, \mathrm{d}V = \int_{S} \rho \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}$$
$$\implies \int_{V} \mathrm{d}V = \int_{S} \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}$$

The left side is simply V. Then, taking  $\frac{\mathrm{d}}{\mathrm{d}t}$  on both sides gives

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{S}$$
$$= \int \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{S}$$
$$= \int \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$$

The second result is a generalization of the previous result. The total mass is

$$M = \int_V \rho \,\mathrm{d}V = \int_V 
ho oldsymbol{x} \cdot \mathrm{d}oldsymbol{S}$$

We can take  $\frac{d}{dt}$  to see how the mass changes with time. By the chain rule, it follows that

$$\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{V} \rho \,\mathrm{d}V \\ &= \int_{V} \frac{\partial \rho}{\partial t} \,\mathrm{d}V + \int_{\partial V} \rho \frac{\partial \boldsymbol{x}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} \\ &= \int_{V} \frac{\partial \rho}{\partial t} \,\mathrm{d}V + \int_{\partial V} \rho \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} \end{aligned}$$

In words, this equation says that in a time  $\delta t$ , the change in the mass of the fluid has contributions from changes in density as well as how much fluid leaves from open surfaces. Note that setting  $\rho = \text{constant}$  gives us our first equation back.